SHORT PROOF OF RAYLEIGH'S THEOREM AND EXTENSIONS

OLIVIER BERNARDI

Abstract. Consider a walk in the plane made of n unit steps, with directions chosen independently and uniformly at random at each step. Rayleigh's theorem asserts that the probability for such a walk to end at a distance less than 1 from its starting point is $1/(n+1)$. We give an elementary proof of this result. We also prove the following generalization valid for any probability distribution μ on the positive real numbers: if two walkers start at the same point and make respectively m and n independent steps with uniformly random directions and with lengths chosen according to μ, then the probability that the first walker ends farther than the second is $m/(m+n)$.

We consider random walks in the Euclidean plane. Given some real positive random variables $X_1, X_2, \dots, X_n$, we consider a random walk starting at the origin of the plane and made of n steps of respective length $X_1, X_2, \dots, X_n$, with the direction of each step chosen independently and uniformly at random. We denote by $X_1 \oplus X_2 \oplus \cdots \oplus X_n$ the random variable corresponding to the distance between the origin and the end of the walk. This definition is illustrated in Figure 1(a).

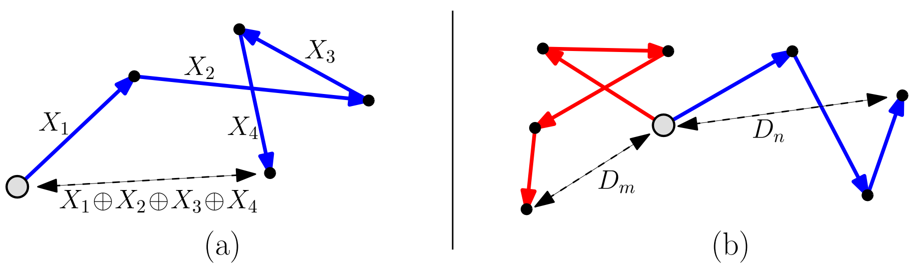

Figure 1. (a) The distance $X_1 \oplus X_2 \oplus X_3 \oplus X_4$ achieved after four steps. (b) Comparing the distances $D_m = m \odot X$ and $D_n = n \odot X$.

For a non-negative real random variable X, we denote by $n \odot X$ the random variable $X_1 \oplus \cdots \oplus X_n$, where $X_1, \dots, X_n$ are independent copies of X. Hence $n \odot X$ represent the final distance from the origin after taking n independent steps of lengths distributed like X and directions chosen uniformly at random. Rayleigh's theorem asserts that if $X = 1$, that is, each step has unit length, then for all $n > 1$,

$$\mathbb{P}(n \odot X < 1) = \frac{1}{n+1}.$$

This theorem was first derived from Rayleigh's investigation of "random flights" in connection with Bessel functions (see [3]) and appears as an exercise in [2, p.104][1]. A simpler proof was given by Kenyon and Winkler as a corollary of their result on branched polymers [1]. The goal of this note is to give an elementary proof of the following generalization of Rayleigh's theorem.

I acknowledge partial support from ANR A3 and European project ExploreMaps.

[1]The exercise calls for developing the requisite Fourier analysis for spherically symmetric functions in order to obtain an identity involving Bessel functions.

Theorem 1. *Let X be a real random variable taking positive values, and let m, n be non-negative integers such that $m + n > 2$. If D_m and D_n are independent random variables distributed respectively like $m \odot X$ and $n \odot X$, then*

$$\mathbb{P}(D_m > D_n) = \frac{m}{m+n}.$$

In words, if two random walkers start at the origin and take respectively m and n independent steps with uniformly random directions and with lengths chosen according to the distribution of X, then the probability that the first walker ends farther from the origin than the second walker is $m/(m+n)$.

Theorem 1 is illustrated in Figure 1(b). Clearly, this extends Rayleigh's theorem which corresponds to the case $m = 1$ and $X = 1$. Our proof of Theorem 1 starts with a lemma based on the fact that the angles of a triangle sum to π.

Lemma 2. *For any random variables A, B, C taking real positive values,*

$$\mathbb{P}(A > B \oplus C) + \mathbb{P}(B > A \oplus C) + \mathbb{P}(C > A \oplus B) = 1. \tag{1}$$

Proof. By conditioning on the values of the random variables A, B, C, it is sufficient to prove (1) in the case where A, B, C are non-random positive constants, and the randomness only resides in the directions of the steps. Now we consider two cases. First suppose that one of the lengths A, B, C is greater than the sum of the two others. In this case, one of the probabilities appearing in (1) is 1 and the others are 0, hence the identity holds. Now suppose that none of the lengths A, B, C is greater than the sum of the two others. In this case, there exists a triangle T with side lengths A, B, C. The triangle T is shown in Figure 2. The probability $\mathbb{P}(A > B \oplus C)$ is equal to α/π, where α is the angle between the sides of length B and C in the triangle T (because $A > B \oplus C$ if and only if the angle between the step of length B and the step of length C is less than α in absolute value). Summing this relation for the three probabilities appearing in (1) gives

$$\mathbb{P}(A > B \oplus C) + \mathbb{P}(B > A \oplus C) + \mathbb{P}(C > A \oplus B) = \frac{\alpha + \beta + \gamma}{\pi} = 1.$$

where α, β, γ are the angles appearing in Figure 2. □

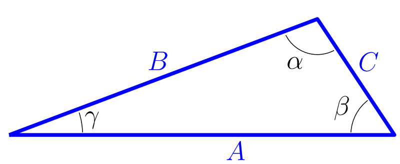

FIGURE 2. The triangle T with side lengths A, B, C.

We now complete the proof of Theorem 1. Let $s = m + n$ and let $D_0, D_1, \ldots, D_s$ be independent random variables distributed respectively like $0 \odot X, 1 \odot X, \ldots, s \odot X$. We denote $p_i = \mathbb{P}(D_i > D_{s-i})$ and want to prove $p_m = m/s$. Let i, j, k be positive integers summing to s. Applying Lemma 2 to $A = D_i$, $B = D_j$, $C = D_k$ gives $p_i + p_j + p_k = 1$. Moreover, $p_k = 1 - p_{s-k}$ since $\mathbb{P}(D_k = D_{s-k}) = 0$ (recall that $s > 2$). Thus

$$p_i + p_j = p_{i+j},$$

for all $i, j > 0$ such that $i + j \leq n$. By induction, this implies $i\,p_1 = p_i$ for all $i \in \{1, \ldots, n\}$. In particular $p_1 = p_s/s = 1/s$, and $p_m = m\,p_1 = m/s$. This concludes the proof of Theorem 1.

Acknowledgments: I thank Peter Winkler for extremely stimulating discussions.

References

[1] R. Kenyon and P. Winkler. Branched polymers. *Am. Math. Monthly*, 7:612–628, 2009.
[2] F. Spitzer. *Principles of random walk*. Van Nostrand, 1964.
[3] G.N. Watson. *A treatise on the Theory of Bessel functions*. Cambridge U. Press, 1944.